ASSIGNMENT 6

Exercise 1. (Converse for Gaussian channel $X \to X+Z$ with $Z \sim \mathcal{N}(0,\sigma^2)$). Consider any $(2^{nR}, n)$ code that satisfies the power constraint, that is,

$$\frac{1}{n}\sum_{i=1}^{n} x_i(w)^2 \leq P,$$

for $w = 1, 2, \ldots, 2^{nR}$. Let P_i denote the average power of the i-th column of the codebook, that is,

$$P_i = \frac{1}{2^{nR}}\sum_{w} x_i(w)^2.$$

a. Let W be distributed uniformly over $\{1, 2, \ldots, 2^{nR}\}$. Let $\widehat{W}$ be the estimate of W based on Y^n. Let $\epsilon_n \to 0$ as probability of error for the code goes to 0. Justify the steps with labels on the equality or the inequality signs.

$$\begin{aligned} nR = H(W) &= I(W;\widehat{W}) + H(W|\widehat{W}) \\ &\overset{(a)}{\leq} I(W;\widehat{W}) + n\epsilon_n \\ &\overset{(b)}{\leq} I(X^n;Y^n) + n\epsilon_n \\ &= h(Y^n) - h(Y^n|X^n) + n\epsilon_n \\ &\overset{(c)}{=} h(Y^n) - h(Z^n) + n\epsilon_n \\ &\overset{(d)}{\leq} \sum_{i=1}^{n} h(Y_i) - h(Z_i) + n\epsilon_n. \end{aligned}$$

b. Calculate $\mathbb{E}[Y_i^2]$ and deduce that

$$nR \leq \frac{1}{2}\sum_{i=1}^{n} \log\left(1 + \frac{P_i}{\sigma^2}\right) + n\epsilon_n.$$

c. Prove that $\frac{1}{n}\sum_{i=1}^{n} P_i \leq P$.

d. Use Jensen's inequality to conclude that

$$R \leq \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

for any code for which the probability of error goes to 0.

Solution. See CT Section 9.2. □

Exercise 2. (List decoding Fano's inequality) Consider discrete random variables X and Y with X taking values in the set $\{0,1\}^k$. Upon observing Y we produce a list $L(Y)$ of size 2^ℓ such that

$$\mathbb{P}(X \in L(Y)) \geq 1 - \varepsilon.$$

Show that

$$H(X|Y) \leq \varepsilon k + (1-\varepsilon)\ell + 1.$$

Hint – Define the random variable $T = \mathbf{1}_{\{X \in L(Y)\}}$. Expand $H(X,Y,T)$ using chain rule.

Solution. Let $T = \mathbf{1}_{\{X \in L(Y)\}}$. Since T is a function of X and Y, we have

$$\begin{aligned} H(X,Y) &= H(X,Y,T) \\ &= H(T) + H(Y|T) + H(X|Y,T). \end{aligned}$$

Since $H(X,Y) = H(Y) + H(X|Y)$ and $H(Y|T) \leq H(Y)$, we have

$$\begin{aligned} H(X|Y) &\leq H(T) + H(X|Y,T) \\ &= H(T) + H(X|Y,L(Y),T) \\ &\leq H(T) + H(X|L(Y),T), \end{aligned}$$

since $L(Y)$ is a function of Y. Now,

$$H(X|L(Y),T=1) \leq \ell,$$

and

$$H(X|L(Y),T=0) \leq k - \ell,$$

and hence

$$H(X|L(Y),T) \leq \mathbb{P}[T=1]\ell + \mathbb{P}[T=0](k-\ell).$$

The result follows by observing that $\mathbb{P}[T=1] \leq 1$, $\mathbb{P}[T=0] \leq \varepsilon$, and $H(T) \leq 1$. □

Exercise 3. (Shearer's lemma) Shearer's lemma is a generalization of the basic inequality

$$H(X_1,\ldots,X_n) \leq \sum_{i=1}^n H(X_i).$$

For $S \subseteq [n] = \{1,2,\ldots\}$, we write $X_S = (X_i : i \in S)$.

a. Prove the following Lemma: Let $X_1,\ldots,X_n$ be random variables. Let $S_1,\ldots,S_m \subseteq [n]$ be subsets such that each $i \in [n]$ belongs to at least k sets. Then,

$$kH(X_1,\ldots,X_n) \leq \sum_{j=1}^m H(X_{S_j}).$$

Solution. Let $S_j = \{i_1,\ldots,i_{s_j}\}$ with $i_1 < \ldots < i_{s_j}$. Then,

$$\begin{aligned} H(X_{S_j}) &= H(X_{i_1}) + H(X_{i_2}|X_{i_1}) + \ldots + H(X_{i_{s_j}}|X_{i_1},\ldots,X_{i_{s_j-1}}) \\ &\geq H(X_{i_1}|X_1,\ldots,X_{i_1-1}) + H(X_{i_2}|X_1,\ldots,X_{i_2-1}) + \ldots + H(X_{i_{s_j}}|X_1,\ldots,X_{i_{s_j-1}}). \end{aligned}$$

Sum the left side over $j=1$ to m to obtain $\sum_{i=1}^m H(X_{S_j})$. Since each $i \in [n]$ belongs to at least k sets from $S_j, j=1,\ldots,m$, the sum of the right side over $j=1$ to m is equal to at least k times the sum $\sum_{i=1}^n H(X_i|X^{i-1})$, whereby the result follows. □

b. Suppose n distinct points in $\mathbb{R}^3$ have n_1 distinct projections on the XY-plane, n_2 distinct projections on the XZ-plane, and n_3 distinct projections on the YZ-plane. For two different points, since all three projections cannot be the same, we have $n \leq n_1 n_2 n_3$. Using Shearer's lemma, show that

$$n \leq \sqrt{n_1 n_2 n_3}.$$

Hint – Let $P = (X_1, X_2, X_3)$ be one of the n points picked uniformly at random. Then, $P_1 = (X_1, X_2)$, $P_2 = (X_1, X_3)$, and $P_3 = (X_2, X_3)$ are its three projections.

Solution. By Shearer's lemma, we have

$$2H(P) \leq H(P_1) + H(P_2) + H(P_3).$$

The results follows since $H(P) = \log n$ and $H(P_i) \leq \log n_i, i = 1, 2, 3$. □